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1994 J. Phys. A: Math. Gen. 27 6533

(http://iopscience.iop.org/0305-4470/27/19/023)

J. Phys. A: Math. Gen. 27 (1994) 6533 6538. Printed in the UK

A q-analogue of the supersymmetric oscillator and its q-supercoherent states

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Received 3 May 1994, in final form 28 June 1994

Abstract. A q-analogue of the supersymmetric oscillator is constructed out of q-boson and ordinary fermion creation and annihilation operators. q-supercoherent states are explicitly obtained for the q-deformed supersymmetric oscillator. They are shown to be eigenstates of both q-boson and fermion annihilation operators and to satisfy a completeness relation. The representation of the q-deformed superalgebra in super Bargmann–Fock space is also discussed by means of the q-supercoherent states.

The coherent-state method [1] is a very powerful and elegant method for the study of algebra (or group) representations. Recently, this method has been used for the study of superalgebras [2–4] and q-deformed superalgebras [5, 6]. In [7], a q-analogue of the supersymmetric oscillator and corresponding q-superalgebra were constructed out of q-boson and q-fermion creation and annihilation operators. Because of the equivalence of both the q-deformed fermion and the ordinary fermion [8, 9], therefore, it is worth reconstructing the q-deformed supersymmetric oscillator and q-superalgebra by using q-boson and ordinary fermion creation and annihilation operators. Furthermore, q-supercoherent states as well as super Bargmann–Fock space have been introduced for the study of the q-supersymmetric oscillator and corresponding q-superalgebra. It can be seen that some new results obtained here are different from those in [7].

In the ordinary supersymmetric theory the superalgebra [10] is generated by H, Q_+ and Q_-, where H (Hamiltonian) is the even generator and $Q_\pm$ are the odd generators of the superalgebra. They satisfy the following relations

$$\{Q_+, Q_-\} = H \qquad [Q_\pm, H] = 0. \tag{1}$$

In order to construct the q-deformed superalgebra for the q-analogue of the supersymmetric oscillator, first of all we have to introduce the q-deformed boson oscillator [11, 12], whose algebra $\{a_q, a_q^\dagger, N_{\mathrm{B}}\}$ are defined by

$$[a_q, a_q^\dagger] = [N_{\mathrm{B}} + 1] - [N_{\mathrm{B}}] \tag{2a}$$

$$[N_{\mathrm{B}}, a_q^\dagger] = a_q^\dagger \qquad [N_{\mathrm{B}}, a_q] = -a_q \tag{2b}$$

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where

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}}. \tag{3}$$

In addition we introduce further the ordinary fermion creation and annihilation operators $f^\dagger$ and f, respectively, and the fermion number operator $N_f=f^\dagger f$. As is well known, they satisfy the following relations

$$\{f,f^\dagger\}=1 \qquad f^{\dagger 2}=f^2=0 \tag{4a}$$

$$[N_f,f^\dagger]=f^\dagger \qquad [N_f,f]=-f. \tag{4b}$$

Now let us discuss the q-supersymmetric oscillator. Defining the odd generators $Q_\pm$ by

$$Q_+=a_q f^\dagger \qquad Q_-=a_q^\dagger f \tag{5}$$

which convert a q-boson into an ordinary fermion and vice versa, respectively, the Hamiltonian H of the q-supersymmetric oscillator may be written as

$$H=\{Q_+,Q_-\}=[N_B]+([N_B+1]-[N_B])N_f. \tag{6}$$

It is obvious that in the $q=1$ case the Hamiltonian H given by (6) coincides with that of the supersymmetric oscillator [3]. Along with operators N_B and N_f, some more relations can be also derived easily

$$[N_B,N_f]=0 \tag{7}$$

$$[Q_\pm,N_B]=\pm Q_\pm \qquad [Q_\pm,N_f]=\mp Q_\pm \tag{8}$$

we have thus obtained a q-superalgebra, defined by the relations (6), (7) and (8). It is seen that this q-superalgebra is generated by the set $\{N_B, N_f, Q_+, Q_-\}$. Even generators N_B and N_f generate two commuting U(1) groups, while the odd generators $Q_\pm$ contain both N_B and N_f in their anticommutator. Since the odd generators $Q_\pm$ are nilpotent, i.e.

$$Q_+^2=Q_-^2=0 \tag{9}$$

the commutation relation $[Q_\pm,H]=0$ is naturally satisfied although $Q_\pm$ do not commute with both N_B and N_f. It means that the Hamiltonian H of the q-supersymmetric oscillator is invariant under the q-superalgebra. This is the same with the $q=1$ case [3].

In order to define q-supercoherent states and to discuss their characterization it is necessary to give a representation space. The natural choice is the super Fock space

$$\mathscr{F}=\{|n_B\rangle\otimes|n_F\rangle=|n_B,n_F\rangle|(n_B=0,1,2,\ldots;n_F=0,1)\} \tag{10}$$

with the eigenstates of number operators N_B and N_f as basic vectors

$$N_B|n_B,n_F\rangle=n_B|n_B,n_F\rangle \qquad N_f|n_B,n_F\rangle=n_F|n_B,n_F\rangle. \tag{11}$$

The fermionic sector is generated by $|n_B,1\rangle$ for all values of n_B; the q-bosonic one by $|n_B,0\rangle$. Then starting from the q-boson vacuum state $|0,n_F\rangle$ defined by $a_q|0,n_F\rangle=0$ one can obtain the n_B-quanta eigenstate explicitly given by

$$|n_B,n_F\rangle=\frac{(a_q^\dagger)^{n_B}}{\sqrt{[n_B]!}}|0,n_F\rangle \tag{12}$$

with

$$a_q^{\dagger}|n_{\mathrm{B}}, n_{\mathrm{F}}\rangle = \sqrt{[n_{\mathrm{B}}+1]}\,|n_{\mathrm{B}}+1, n_{\mathrm{F}}\rangle \tag{13a}$$

$$a_q|n_{\mathrm{B}}, n_{\mathrm{F}}\rangle = \sqrt{[n_{\mathrm{B}}]}\,|n_{\mathrm{B}}-1, n_{\mathrm{F}}\rangle \tag{13b}$$

and

$$f^{\dagger}|n_{\mathrm{B}}, 0\rangle = |n_{\mathrm{B}}, 1\rangle \qquad f|n_{\mathrm{B}}, 0\rangle = 0 \tag{14a}$$

$$f^{\dagger}|n_{\mathrm{B}}, 1\rangle = 0 \qquad f|n_{\mathrm{B}}, 1\rangle = |n_{\mathrm{B}}, 0\rangle \tag{14b}$$

where

$$[n_{\mathrm{B}}]! = [n_{\mathrm{B}}]\cdot[n_{\mathrm{B}}-1]\ldots[2]\cdot[1]. \tag{15}$$

Now let us define q-supercoherent states as

$$|z, \eta) = e_q(za_q^{\dagger})(1-\eta f^{\dagger})|0, 0\rangle \tag{16}$$

where $e_q(za_q^{\dagger})$ is the q-exponential operator defined by

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]!} \tag{17}$$

and z is a c-number (even Grassmann number) while η is an a-number (odd Grassmann number) [13]. Using an abbreviation

$$|z, n_{\mathrm{F}}) = \sum_{n_{\mathrm{B}}=0}^{\infty}\frac{z^{n_{\mathrm{B}}}}{\sqrt{[n_{\mathrm{B}}]!}}|n_{\mathrm{B}}, n_{\mathrm{F}}\rangle \tag{18}$$

q-supercoherent states may be rewritten as

$$|z, \eta) = |z, 0) - \eta|z, 1) \tag{19}$$

where the q-bosonic and fermionic sectors $|z, 0)$ and $|z, 1)$ of q-supercoherent states $|z, \eta)$ have to be regarded as c- and a-type states, respectively. The q-supercoherent states defined by (16) are neither unity normalized nor orthogonal. Actually, we have

$$\begin{aligned}(z_1, \eta_1|z_2, \eta_2) &= (z_1, 0|z_2, 0) + \bar{\eta}_1\eta_2(z_1, 1|z_2, 1)\\ &= (1+\bar{\eta}_1\eta_2)e_q(\bar{z}_1 z_2).\end{aligned} \tag{20}$$

Further, direct calculation shows that the q-supercoherent states are eigenstates of both the q-bosonic and fermionic annihilation operators a_q and f. Indeed, we have

$$a_q|z, \eta) = z|z, \eta) \qquad f|z, \eta) = \eta|z, \eta). \tag{21}$$

Note that the completeness relation for the q-bosonic coherent states can be written as [14]

$$\int |z\rangle_{q\,q}\langle z|\,\mathrm{d}\mu(z) = \sum_{n=0}^{\infty}|n\rangle\langle n| \qquad |z\rangle_q = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n\rangle \tag{22}$$

where

$$\mathrm{d}\mu(z) = \frac{1}{2\pi}\,e_q(-|z|^2)\,\mathrm{d}_q|z|^2\,\mathrm{d}(\arg z) \tag{23}$$

is the q-integration measure, the completeness relation for the q-supercoherent states may be written in a matrix form as

$$\int |z, \eta) D(z, \eta)(z, \eta| = I \tag{24}$$

where

$$|z, \eta) = (|z, 0) \quad -\eta|z, 1)) \tag{25a}$$

$$(z, \eta| = \begin{pmatrix} (z, 0| \\ -(z, 1|\bar{\eta} \end{pmatrix} \tag{25b}$$

and the integration measure is denoted by a square matrix

$$D(z, \eta) = \begin{pmatrix} \mathrm{d}\mu(z) & 0 \\ 0 & \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{d}\mu(z) \end{pmatrix}. \tag{25c}$$

In fact, using (22), we have

$$\begin{aligned} \int |z, \eta) D(z, \eta)(z, \eta| &= \int \{|z, 0)(z, 0| + |z, 1)(z, 1|\eta\bar{\eta}\,\mathrm{d}\bar{\eta}\,\mathrm{d}\eta\}\,\mathrm{d}\mu(z) \\ &= \int \{|z, 0)(z, 0| + |z, 1)(z, 1|\}\,\mathrm{d}\mu(z) \\ &= \sum_{n_B} \{|n_B, 0\rangle\langle n_B, 0| + |n_B, 1\rangle\langle n_B, 1|\} \\ &= I. \end{aligned} \tag{26}$$

Using the q-supercoherent states defined by (16), it is not difficult to introduce the super Bargmann–Fock representation, namely

$$\begin{aligned} |n_B, n_F\rangle \to x_{n_B, n_F}(z, \eta) &= (\bar{z}, \bar{\eta}|n_B, n_F\rangle \\ &= (\bar{z}, 0|n_B, n_F\rangle + \eta(\bar{z}, 1|n_B, n_F\rangle \\ &= (\delta_{n_F,0} + \eta\delta_{n_F,1}) \frac{z^{n_B}}{\sqrt{[n_B]!}} \end{aligned} \tag{27a}$$

$$\begin{aligned} |\psi\rangle = \sum_{n_B} \{c_{n_B}|n_B, 0\rangle + \mathrm{d}_{n_B}|n_B, 1\rangle\} \to \psi(z, \eta) &= (\bar{z}, \bar{\eta}|\psi\rangle = \psi_0(z) + \eta\psi_1(z) \\ &= \sum_{n_B} \{c_{n_B} + \eta\,\mathrm{d}_{n_B}\} \frac{z^{n_B}}{\sqrt{[n_B]!}} \end{aligned} \tag{27b}$$

where, in general, $\psi_0(z) = (\bar{z}, 0|\psi\rangle$ and $\psi_1(z) = (\bar{z}, 1|\psi\rangle$ are analytic functions of complex variable z. In the super Bargmann–Fock space, the following expressions for the generators of q-superalgebra can easily be derived

$$\Gamma(Q_+)=\eta\frac{\mathrm{d}}{\mathrm{d}_q z}\qquad \Gamma(Q_-)=z\frac{\mathrm{d}}{\mathrm{d}\eta} \tag{28a}$$

$$\Gamma(N_\mathrm{B})=z\frac{\mathrm{d}}{\mathrm{d}z}\qquad \Gamma(N_\mathrm{f})=\eta\frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{28b}$$

Actually, we have, for example, from (27a, b)

$$\begin{aligned}(\bar z,\bar\eta|Q_+|\psi\rangle&=(\bar z,\bar\eta|a_q f^\dagger|\psi\rangle\\ &=\sum_{n_\mathrm{B}}c_{n_\mathrm{B}}\sqrt{[n_\mathrm{B}]}(\bar z,\bar\eta|n_\mathrm{B}-1,1)\\ &=\eta\sum_{n_\mathrm{B}}c_{n_\mathrm{B}}[n_\mathrm{B}]\frac{z^{n_\mathrm{B}-1}}{\sqrt{[n_\mathrm{B}]!}}\\ &=\eta\frac{\mathrm{d}}{\mathrm{d}_q z}\psi(z,\eta).\end{aligned} \tag{29}$$

Furthermore, the inner product can be written by means of (24) as follows

$$\langle\varphi|\psi\rangle=\int\langle\varphi|\bar z,\bar\eta)D(\bar z,\bar\eta)(\bar z,\bar\eta|\psi\rangle. \tag{30}$$

In addition, we can also prove that the Hermiticity properties $Q_\pm^\dagger=Q_\mp$, $N_\mathrm{B}^\dagger=N_\mathrm{B}$ and $N_\mathrm{f}^\dagger=N_\mathrm{f}$ are entirely retained with respect to the inner product given by (30), i.e.

$$\left(\eta\frac{\mathrm{d}}{\mathrm{d}_q z}\right)^\dagger=z\frac{\mathrm{d}}{\mathrm{d}\eta}\qquad \left(z\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^\dagger=\eta\frac{\mathrm{d}}{\mathrm{d}_q z} \tag{31a}$$

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^\dagger=z\frac{\mathrm{d}}{\mathrm{d}z}\qquad \left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^\dagger=\eta\frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{31b}$$

For example, using (30), we get

$$\begin{aligned}\langle\varphi|Q_+|\psi\rangle&=\int(\bar\varphi_0(z)\quad \bar\eta\bar\varphi_1(z))D(\bar z,\bar\eta)\left(\eta\frac{\mathrm{d}}{\mathrm{d}_q z}\right)\begin{pmatrix}\psi_0(z)\\ \eta\psi_1(z)\end{pmatrix}\\ &=\int(\bar\varphi_0(z)\quad \bar\eta\bar\varphi_1(z))D(\bar z,\bar\eta)\begin{pmatrix}0\\ \eta\dfrac{\mathrm{d}}{\mathrm{d}_q z}\psi_0(z)\end{pmatrix}\\ &=\int\bar\varphi_1(z)\frac{\mathrm{d}}{\mathrm{d}_q z}\psi_0(z)\,\mathrm{d}\mu(z).\end{aligned} \tag{32}$$

On the other hand, we have

$$\int\overline{\left(z\frac{\mathrm{d}}{\mathrm{d}\eta}\right)}(\bar\varphi_0(z)\quad \bar\eta\bar\varphi_1(z))D(\bar z,\bar\eta)\begin{pmatrix}\psi_0(z)\\ \eta\psi_1(z)\end{pmatrix}=\int\bar z\bar\varphi_1(z)\psi_0(z)\,\mathrm{d}\mu(z). \tag{33}$$

Using the q-analogue of Euler's formula for $\Gamma(x)$ in [14], it is not difficult to prove that

$$\int \bar{z}\bar{\varphi}_1(z)\psi_0(z)\,\mathrm{d}\mu(z) = \int \bar{\varphi}_1(z)\frac{\mathrm{d}}{\mathrm{d}_q z}\psi_0(z)\,\mathrm{d}\mu(z) \tag{34}$$

for any analytic functions $\varphi_1(z)$ and $\psi_0(z)$. It means that

$$\left(\eta\frac{\mathrm{d}}{\mathrm{d}_q z}\right)^{\dagger} = z\frac{\mathrm{d}}{\mathrm{d}\eta}.$$

It is obvious that the success in super Bargmann–Fock representation follows from the completeness relation, which belongs to the entire super Fock space. As far as I know, there was a similar completeness relation for supercoherent states in [2], but its identity operator acts in the space of even states only.

Finally, we have to point out that the q-deformed fermion creation and annihilation operators introduced by Parthasarathy and Viswanathan in [7] are not necessary to satisfy (4a), so that not only the n ($n>1$) q-fermion states may be defined, but also, in general, the odd generators $Q_\pm$ in [7] do not satisfy (9). Therefore, the q-deformed superalgebra defined by (6), (7) and (8) are explicitly different from the one defined by (21) in [7]. If we assume that the q-deformed fermion creation and annihilation operators have to satisfy (4a), which arises from Pauli's exclusion principle, the q-deformed fermion will be equivalent to the ordinary one [8, 9]. That is why we reconstruct the q-supersymmetric oscillator and q-superalgebra by using q-boson and ordinary fermion creation and annihilation operators.

In addition, it is worth noticing that the q-supercoherent states defined by (16) are nothing but a natural and reasonable extension of q-oscillator coherent states. In fact, the q-bosonic sector $|z, 0)$ in (19) is the same as the q-analogue of the Heisenberg–Weyl (HW) coherent states introduced in [15, 16].

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